Study of a toy-model for the analysis of stochastic gravitational wave background

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Abstract
The aim of this article is to introduce the method used in order to extract the gravitational wave stochastic background from detectors signals. For this, we study a toy-model which can compute the analysis thanks to two different methods : Cross-Correlation and Optimal-Filtering. The obtained results are plots of the signal-to-noise ratio as function of time or amplitude of the Gravitational Wave Background signal compared to the detector noises. Then we confront those average evolutions to the analytical previsions.

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Contents

Introduction

The General Relativity theory predicts that an accelerated mass will disturb the curvature of space-time. Those disturbances are the so-called Gravitational Waves (GW).

There are multiple types of sources which are able to produce GW but those directly detectable are produced by super-massive objects like spinning neutrons stars and mergers of Binary Black Holes (BBH), Binary Neutrons Stars (BNS) or White Dwarfs (WD).
There is another type of signal called Gravitational Waves Background (GWB) corresponding to the sum of the individually unsolvable events. It is constituted by an astrophysical component (too far or intrinsically too weak events) and a cosmological component (primordial gravitational waves). This last component can be compared to the Cosmological Microwave Background (CMB) in the sense that both of them are relic radiations. This cosmological component is relevant for the study of early univers because it was emitted $10^{-30}s$ after the Big Bang while the earliest signal currently known is the CMB emitted approximately 380 000 years after the Big Bang.

The system of detection is currently composed of three interferometers called LIGO Hanford, LIGO Livingston and Advanced-Virgo. They are all based on Michelson interferometer principle. Thanks to those detectors, in 2014, the signal GW15092014 confirmed the existence of this type of wave.
Currently, the three detectors increase their sensibilities but as efficient as they are, the noises of those detectors is always stronger than the GWB signal.

In the following we base ourselves a toy-model developed by Joseph Romano for L'Ecole des Houches [1]. This approach is very simplified and cannot be applied to real data but let us understand the ins and the outs of the methods used.

1. Gravitational Wave Background

The GWB is a signal presenting statistical properties composed in the superposition of a large number of unresolved sources. It is formed of multiple GWB components associated to different type of sources. They may be of astrophysical origin (BBH mergers, BNS mergers, ...) or of cosmological origin (primordial gravitational waves). We can consider the different components individually and point out the specificities of each. All of them are currently inaccessible but the cosmological one is expected to be far weaker than the astrophysical one. In addition to inform us on several parameters about celestial bodies in the universe, the study of the astrophysical component can allow us to subtract it from the overall signal and hope to access the cosmological component.

1.1 Statistical properties

We assume that, for the simulation of the GWB, three parameters can be statistically described : the arrival time of an event, its amplitude and the direction from which the signal comes from. The arrival time of events it totally random in the way that there are no preferred values for an event to happened. The amplitude of events is relative to the mass of merging bodies. But we can set an average value around which the amplitude fluctuate. So we can use a uniform distribution to randomly determine those two first parameters.
The last parameter is the direction of the source relatively to the detector. The angular distribution of GWB amplitude can be compared to the cosmological microwave background one in the sense that it is approximately isotropically distributed. If we would considering WD mergers, we would see an excess of events in the directions of the galaxy equatorial plane.

1.2 Type of sources

Leading to different types of stochastic backgrounds, the different sources are recognizable thanks to the detected signal and their associate power frequency spectrum. This implies a precise terminology about them.

1.2.1 White noise stochastic background

This first type of signal is the simplest one. The white noise GWB is a simplification which results in a uniformly distribution of frequencies that make up the signal. (See Figure 1)

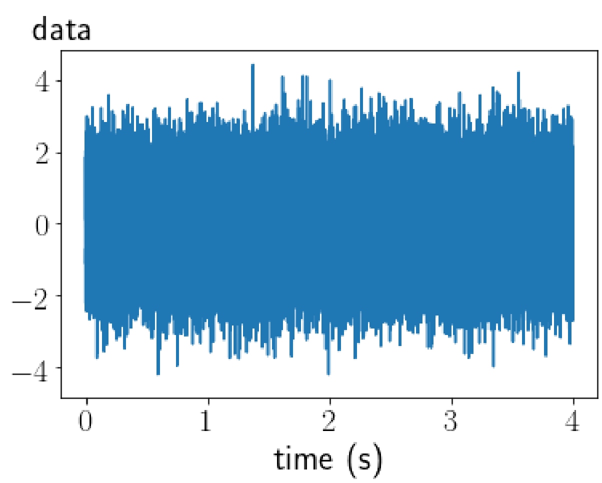

Figure 1. Simulation of a white noise

1.2.2 Pop-corn stochastic background

Here we concentrate on the GWB formed using only the BBH mergers component. Those merger lead to chirp signals resulting in an oscillation which increase in frequency and in amplitude until merging.
We can estimate the number of BBH mergers between one per minute to a few per hour [1], [2]. The average duration of the merger being about one second [1], by comparing those values, we can see that the mergers duration are much smaller than the average time between two events. Based on that, we should see ”peaks” separated by periods of silence when plotting this signal. This characteristic aspect gives the name of ”Pop-corn” like signal to the BBH component of GWB.
The simulation of 10 BBH merger in 10s is presented on the figure 2.

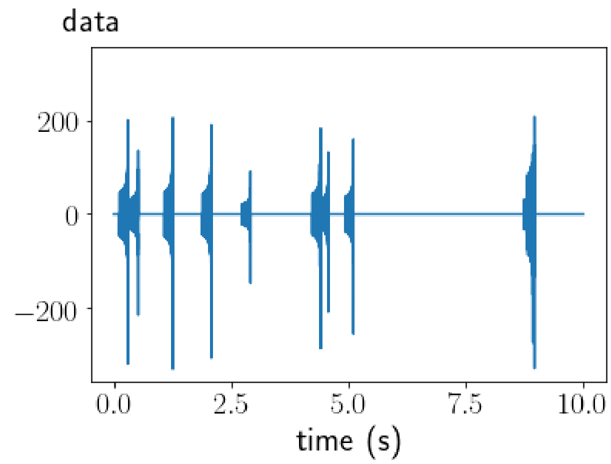

Figure 2. Simulation of pop-corn stochastic background (BBH mergers) for 10 events in 10 seconds

1.2.3 Confusion-limited stochastic background

The last component is the BNS one. For BNS systems, the average time between two mergers is about 15 seconds [1] and the duration of a merger is around 100 seconds [1]. We easily understand that the BNS merger signals will overlap one on each other and give to the stochastic background the shape of a continuous signal. This is called ”Confusion-limitied” stochastic background and refer to the BNS component of GWB. (See figure 3)

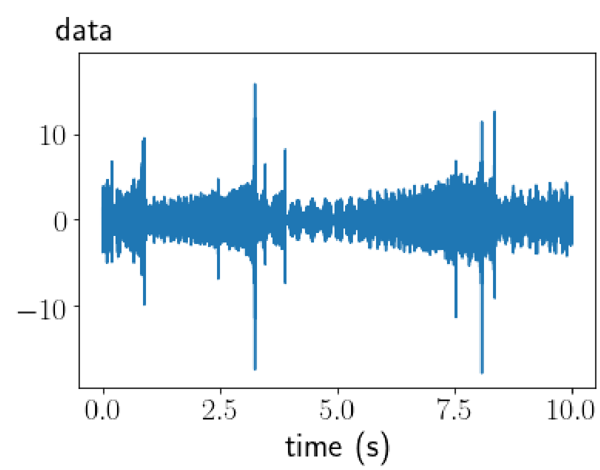

Figure 3. Simulation of confusion-limited stochastic background (BNS mergers) for 10 events in 10 seconds

1.2.4 Associated power spectrum

The power spectrum expected for the white noise stochastic background should be uniformly distributed in frequency. We can see in figure 4 that it fluctuates around a constant value.

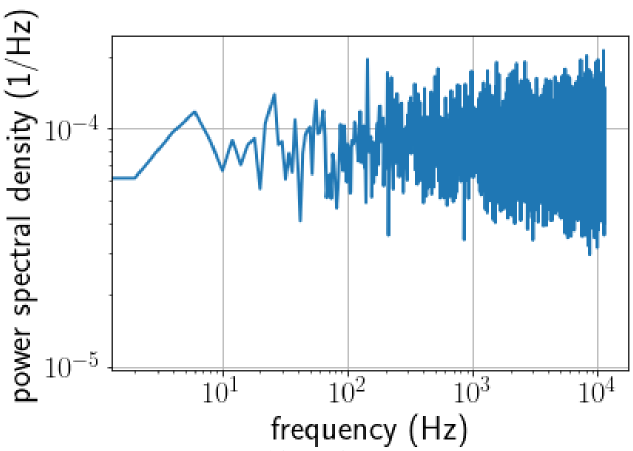

Figure 4. White noise power spectrum

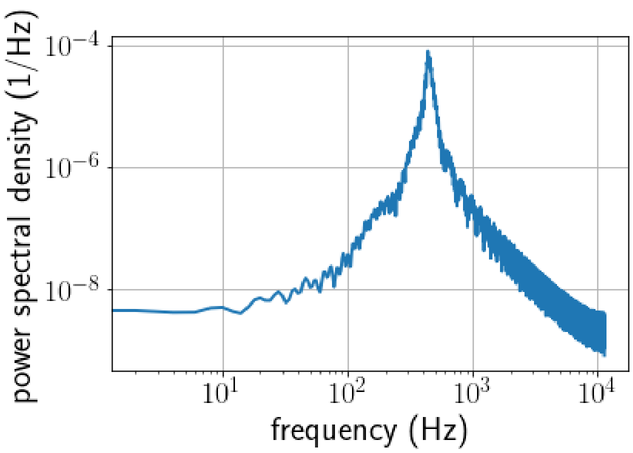

Figure 6. Ring-down power spectrum

The BBH and BNS power spectrum have the same shape. The power distribution can be approximated with a function $H(f^{-\frac{7}{3}})$ (see figure 5). Here just the descending right part of the plot is relevant since the left is just artifacts from the Fourier transform.

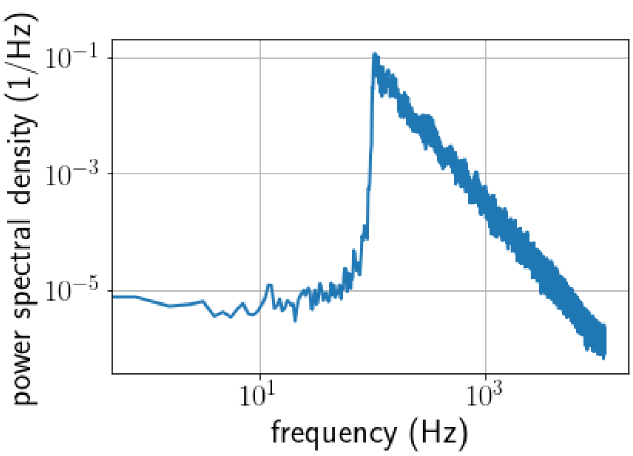

Figure 5. BBH Pop-corn / BNS Confusion-limited power spectrum

On figure 6 is represented the power spectrum of the ring-down associate to the BBH relaxation. After merging, the new Black Hole (BH) formed is spinning and its surface is not at equilibrium meaning that this surface vibrate. This corresponds to the BH quasi-normal modes of vibrations. Actually, those vibrations are composed of a fundamental and harmonics frequencies but by keeping only the most powerful vibration (the fundamental), we can plot a power spectrum showing a maximum at the frequency of the fundamental.

2. Methods

In this part, we explain two methods allowing to extract a signal smaller than the noise of the detector measuring it.
The key point is to make a correlation between multiple detectors. The only common component between two detectors is the GWB (assuming that they are properly separated). Based on that, we can apply an analysis of correlation between the data by comparing them with respect to the time or to the frequency (Cross-Correlation and Optimal Filtering, respectively).
This leads to the two methods presented bellow.

2.1 Cross-Correlation

In this part we make assumptions in order to explain the principle of the method. We first consider that the two detectors are on the same geographical localisation with the same orientation so as to not consider any shifts in arrival time or amplitude of signals because of the spatial separation. We also consider that, even if they are on the same localisation, there is no common component between the noises of the detectors.

2.1.1 Signal determination

The noises of the detectors are not known precisely enough to attribute to an excess of signal the presence of GWB. So as to extract the signal, we can apply the Cross-correlation method. We express the data of detector 1 and detector 2 :

$$d_1 = h + n_1 \qquad d_2 = h + n_2 \tag{1}$$

With d_1, d_2 the output signals of detectors, h the common component (GWB signal) and n_1, n_2 the detectors noises.
In average h is not correlated with n_1 or n_2 and n_1 is not correlated with n_2. So if we multiply two samples d_1 and d_2 taken on the same time, in average, we will obtain :

$$< d_1 d_2 > = < h^2 > \tag{2}$$

By applying now this computation to an entire set of data corresponding to a range of time, we can determine the average amplitude of the GWB signal S_{GWB} :

$$< S_{GWB} > = \frac{1}{N}\sum_{i=1}^{N} < d_{1i} d_{2i} > = \frac{1}{N}\sum_{i=1}^{N} < h^2 > \tag{3}$$

With N the number of samples that data contains.

2.1.2 Noise determination

We can take the variance of the GWB signal and make the dependence on the noise S_n explicit :

$$\sigma^2 = < S_{GWB}^2 > - < S_{GWB} >^2 \tag{4}$$

Inserting (3) in (4) and using $d_{1i} = h + n_{1i}$, $d_{2i} = h + n_{2i}$ we can develop the data product following the Isserlis theorem for 4 variables :

$$< abcd > = < ab >< cd > + < ac >< bd > + < ad >< bc > \tag{5}$$

Leading to the following expression :

$$\sigma^2 = \frac{1}{N}(S_1 S_2 + S_h^2) \tag{6}$$

Assuming that the detector output signals S_1 and S_2 are dominated by the noise signals (which is a valid assumption as the GWB is weak compared to the noises) so that $S_1 \simeq S_{n_1}$ and $S_2 \simeq S_{n_2}$ with $S_{n_1} \simeq S_{n_2} \simeq S_n$. Then we finally have:

$$\sigma^2 \simeq \frac{S_n^2}{N} \tag{7}$$

With S_n the overall noise of the two detectors.

2.1.3 Signal-to-noise ratio

We can combine the previous results in order to compute the expression of the signal-to-noise ratio ρ (SNR). If we take :

$$\rho = \frac{S_{GWB}}{\sigma} \tag{8}$$

And replace the value of the standard deviation with the square root of the expression (7), we obtain :

$$\rho = \sqrt{N}\frac{S_{GWB}}{S_n} \tag{9}$$

We notice that the SNR grows as the square root of the number of samples N. But N and the time t are related by the sampling rate which is constant, so finally :

$$\rho \propto \sqrt{t}\frac{S_{GWB}}{S_n} \tag{10}$$

2.1.4 Error calculation

We estimate the error in the calculation of the SNR thanks to the propagation of uncertainty method.
Then we write :

$$\Delta\rho = \rho\left(\left(\frac{\Delta S_{GWB}}{S_{GWB}}\right)^2 + \left(\frac{\Delta S_n}{S_n}\right)^2\right)^{\frac{1}{2}} \tag{11}$$

But we can express [1]:

$$\Delta S_{GWB} = \frac{S_n}{\sqrt{N}} \tag{12}$$

And apply the same method that allow us to derive the expression (6) on :

$$\Delta S_n = \Delta S_{n_1} + \Delta S_{n_2} \tag{13}$$

$$\Rightarrow \Delta S_n = \sqrt{< S_{n_1}^2 > - < S_{n_1} >^2} + \sqrt{< S_{n_2}^2 > - < S_{n_2} >^2} \tag{14}$$

But S_{n_1} and S_{n_2} are Gaussian noises (centered in 0) so $< S_{n_1} > = < S_{n_2} > = 0$. Implying that we can derive from (14) the expression :

$$\Delta S_n = \sqrt{\frac{3S_{n_1}^2}{N}} + \sqrt{\frac{3S_{n_2}^2}{N}} \tag{15}$$

Where S_{n_1} and S_{n_2} are the noises of detectors 1 and 2 (respectively).
By replacing the value of the SNR in the expression (11), we obtain the final expression for the error :

$$\Delta\rho = \frac{N}{S_n^2}\left(\Delta S_h^2 + \frac{S_h^2 \Delta S_n^2}{S_n^4}\right) \tag{16}$$

We see that the error will be dominated by the left term in the parenthesis since the right term is proportionnal to $\frac{S_h^2}{S_n^4}$ with $S_h << S_n$.

2.2 Optimal Filtering

The Optimal Filtering is different from the Cross-correlation is the sense that it is based on a frequency analysis. This method do a cross-correlation between two sets of data in the space of frequency after applying a Fourier Transform to them.
The particularity of this method is that we need to introduce a filter function that maximize the SNR. It depends on the overlap function between the two detectors, power spectra of the GWB signal and detector noises. The overlap function allows to take in account the difference in geographical localisation and orientation between two detectors. It indicates at which frequencies both of the detectors are able to measure a signal together.
The correlation between detectors can be expressed :

$$C(f) = \Gamma(f)S(f) \tag{17}$$

With C the correlated signal between two separated detectors, Γ the overlap function containing the modifications in frequency power that the physical separation and the misalignment implies and S the power spectrum of the signal.

2.3 Simulation

In order to simplify the test of the two methods presented above, we have generated the signals instead of using real data from LIGO / Advanced-Virgo detectors.

2.3.1 Detector noises

The noise of a detector is constituted by multiple factors as the human activity, the photon shot noises which dominates at

high frequencies or the seismic noise which dominates at low frequencies.
In this work we assume that the noise of a detector is Gaussian which means that all the frequencies are equally represented.
Then we generate the simulated detector noise thanks to a uniform distribution assigning for all sample a random amplitude within a certain range.
Those amplitudes are adjusted to have :

$$< S_{n_1}^2 > \approx < S_{n_2}^2 > \approx 1 \tag{18}$$

2.3.2 Stochastic background signal

In the case of white noise GWB signal, the waveform is constructed in the same way as the detector noises, defining for all time a random value between a certain interval.
In case of BNS and BBH GWB signals, we need to randomly define : the number of events, the arrival time of events and their maximal amplitudes. The number of events is selected in a Poisson distribution for a given parameter λ (maximum value of the distribution). We set this λ parameter for each time in order to keep the "density of events" at $2.5 events/s$.
The arrival times of each events are randomly defined using a uniform distribution between the start and the end of the simulation time. The amplitudes of signals are arbitrarily sets to 10 percents of the noise amplitude with a fluctuation of 50 percents.

2.3.3 Combining signals

In order to do the analysis between two simulated output detector signals, we need to combine the generated signal with the noise of the first detector and with the noise of the second detector. This leads to two signals of the form expressed on figure 7. We can note that in this figure are represented two signals which amplitudes compared to the noise are arbitrarily sets.

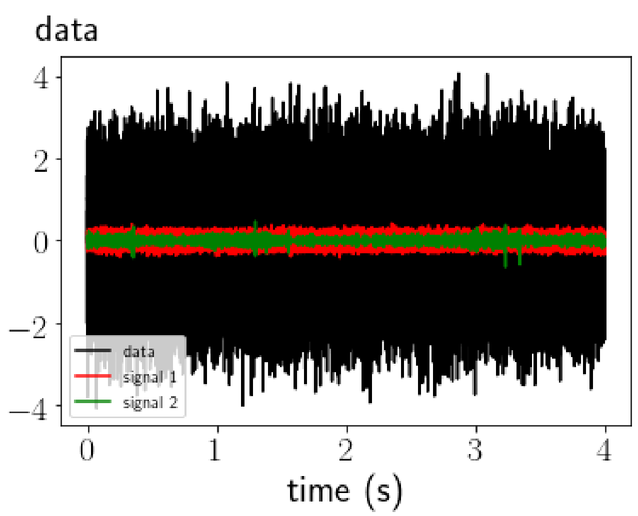

Figure 7. Simulated combined signal : red corresponds to white noise GBW / green corresponds to confusion-limited GWB / black corresponds to detector noise

3. Results

3.1 SNR evolution with time using cross-correlation

In the figure 8 is represented the evolution of the SNR for different type of source (WN / BNS / BBH) for simulation duration ranging from $2s$ to $18s$ using the Cross-Correlation method. We expect an evolution following $\sqrt{N} = \sqrt{22500 \cdot t}$ with N the number of sample, t the time and $f = 22500$ the sampling rate.

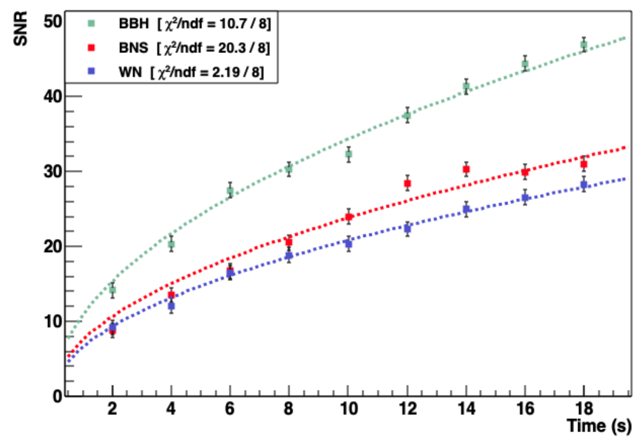

Figure 8. SNR evolutions as functions of time for BBH, BNS and WN signals using cross-correlation method

3.2 SNR evolution with time using optimal-filtering

In the figure 9 is represented the SNR evolution for different sources using the Optimal-Filtering method for simulation duration ranging from $2s$ to $14s$. We again expect an evolution following $\sqrt{N}$.

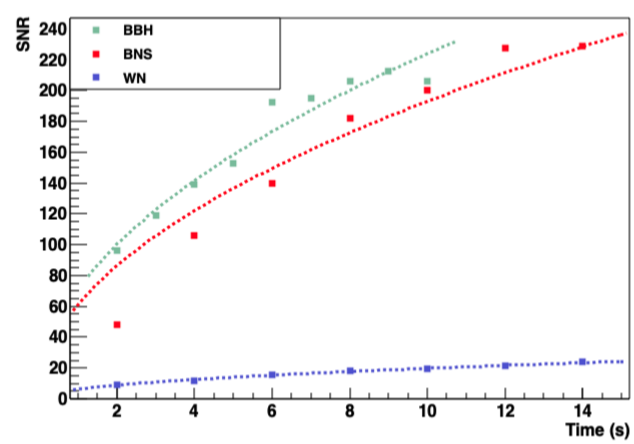

Figure 9. SNR evolutions as functions of time for BBH, BNS and WN signals using optimal-filtering method

3.3 SNR evolution with the amplitude of the gravitational wave signal

In the figure 10 we plot the evolution of the SNR according to the amplitude of the signal compared to the amplitude of the noise. Those values was determined thanks to the Cross-correlation method for a fixed time.

Here, we chose a fit with polynomial of order two passing through the origin. We can point out that the A^2 evolution was expected. In part 2.1 we saw that the amplitude of the signal S_{GWB} grows as $< h^2 >$. But the GWB signal h depend directly on the parameter A which is its amplitude. Then the direct dependance of the SNR in S_{GWB} can be rewritted :

$$\rho \propto A^2 \quad (19)$$

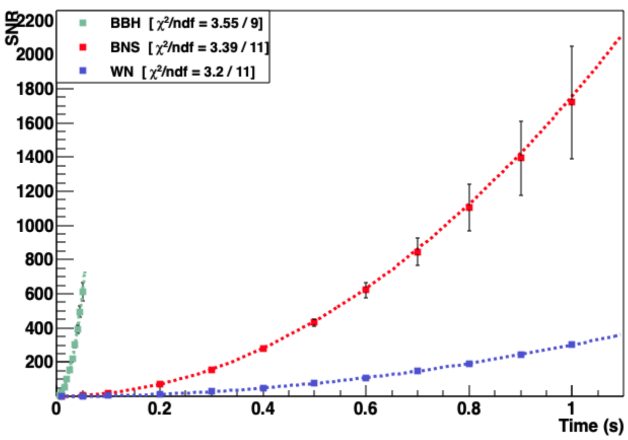

Figure 10. SNR evolutions as functions of the signal amplitude for BBH, BNS and WN for a fixed simulations duration time of $t = 4s$

3.4 SNR evolution with number of detectors

The SNR corresponding to the analysis between N detectors during a time t can be expressed as a function of the SNR resulting from a two detectors analysis. The data amount of a multiple detectors system leads to the exact same situation as if we just measure with two detectors during a longer time. So increasing the number of detectors allows to form $\frac{N(N-1)}{2}$ pairs of two detectors system, which is equivalent to increase the measurement time of a single two detectors system by this factor. From this, we can formally express that measuring thanks to N detectors during a time t corresponds to a measurement with two detectors during a time $\frac{N(N-1)}{2}t$[3]. But reminding that the SNR grows as the square root of the time, for the same measurement time, we can express the associated SNR ρ :

$$\begin{cases} \rho_{3det} = \sqrt{3} \cdot \rho_{2det} \\ \rho_{4det} = \sqrt{6} \cdot \rho_{2det} \\ \rho_{5det} = \sqrt{10} \cdot \rho_{2det} \\ \dots \\ \rho_{Ndet} = \sqrt{\frac{N(N-1))}{2}} \cdot \rho_{2det} \end{cases} \quad (20)$$

The evolution of a N detectors system compared to a two detectors system for the same time of measurement is expressed on figure 11.

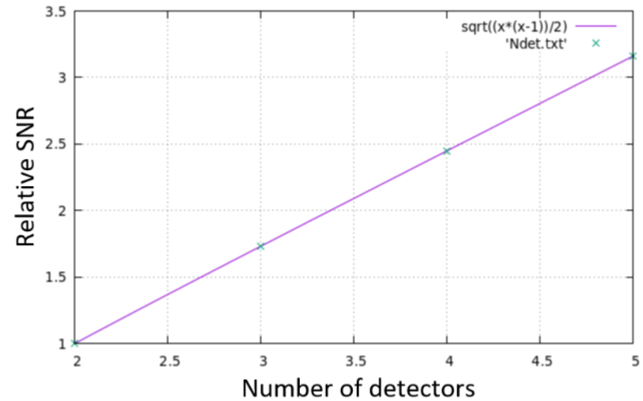

Figure 11. Evolution of the SNR with the number of detectors

Conclusion

In this study, we used a toy-model which aim is to analyse the gravitational wave stochastic background. Thanks to two methods allowing to extract a signal component in a noise (Cross-correlation and Optimal-Filtering), the evolution of the signal-to-noise ratio respectively to two different parameters (time and amplitude of the stochastic background) were plotted.
We see that the results corresponds to the analytical previsions concerning the evolution of the SNR with resepct to the time or the amplitude of the signal. Another interesting result is the evolution of the SNR with the number of detectors. Finally with the construction of the new detectors KAGRA and IndIGO respectively in Japan and in India, there will be five available interferometers. Under the assumption that those interferometers have the same performances, the expected increase in SNR for the measurement of GWB is foreseen to be of the order of $\sqrt{10} \simeq 3$.

The next step is to develop this model in order to work with spatially separated and misaligned detectors. We might also be interested on other parameters such as polarisation of gravitational wave to be able to use real data.

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